

# Chapter 1

## Introduction

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### 1.1 How the Casimir Force was Discovered

Casimir forces are associated with topological constraints on quantum fields. The most famous such effect was predicted in 1948 by Casimir [1], who found that there is an attractive force

$$F = -\frac{\pi^2 \hbar c}{240d^4} \quad (1.1)$$

per unit area between two parallel, uncharged, perfectly conducting plates separated by a distance  $d$ . Casimir derived this force as a consequence of the change in the (infinite) zero-point electromagnetic field energy due to the presence of the plates. Lifshitz [2], taking a more general approach based on electromagnetic fluctuations in thermal equilibrium, obtained Casimir's result as the perfect-conductor limit of the force between two dielectric half-spaces separated by a vacuum. While generally very weak except at very small separations, Casimir effects are of great interest for both theoretical and practical reasons. Their very existence stands in contradiction to the prediction of classical electrodynamics that there should be no force, as is evident from the appearance of  $\hbar$  in (1.1). More practical reasons for the recent interest in Casimir effects are their implications for

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micro-electromechanical (MEMS) devices and other systems in which material components are in close proximity.

Casimir was led to his celebrated formula by some rather down-to-earth considerations on the stability of colloids. In this sense the subject can be traced back to the 1873 doctoral thesis of a struggling thirty-six-year-old graduate student, Johannes van der Waals, who many years later (1910) would be awarded the Nobel Prize in Physics. Van der Waals originally suggested a molecular interaction potential of the form  $V(r) = -Ar^{-1}e^{-Br}$ , where  $r$  is the distance between the molecules and  $A$  and  $B$  are positive constants. It is now well understood, of course, that there are different types of van der Waals forces. The simplest to understand is the orientational (or Keesom) interaction between two molecules with permanent electric dipole moments  $p_1, p_2$ :  $V(r) = -p_1^2 p_2^2 / 3kTr^6$  at temperature  $T$ . The force is attractive because attractive orientations are energetically favored over repulsive ones. There is also an induction (or Debye) interaction between two molecules, one of which has a permanent dipole (or quadrupole) moment. Neither of these interactions is sufficiently general to account for the van der Waals equation of state, which requires an attractive interaction even if the molecules have no permanent electric (or magnetic) moments. For this purpose what is required is a third type of van der Waals interaction, first derived by London [3]. For two identical molecules with polarizability  $\alpha(\omega)$ , this *dispersion force* is

$$V(r) = -\frac{3\hbar}{\pi r^6} \int_0^\infty d\xi \alpha^2(i\xi) \equiv -\frac{C}{r^6}. \quad (1.2)$$

It is called a “dispersion” force because of the appearance of the molecular polarizability, which determines the refractive index  $n(\omega)$  via the relation  $n^2(\omega) = 1 + 4\pi N\alpha(\omega)$  for a sufficiently dilute medium of  $N$  molecules per unit volume. It is important to note that  $\alpha(i\xi)$  is a real number, which follows from the Kramers–Kronig relations between the real and imaginary parts of  $\alpha(\omega)$ .

The dispersion force is, of course, important in many contexts [4], one of them having to do with the stability of certain colloids, such as oil-in-water emulsions, in which interactions between the suspended particles and the molecules of the medium are negligible. Such a colloid is said to be stable if there is no coagulation. An example of a stable colloidal suspension is homogenized milk, in which the suspended particles are fat globules. Homogenization breaks up the globules into pieces sufficiently small that, in addition to other effects, the attractive forces between them are too weak to turn the milk into cream.

Colloidal stability requires that the repulsion between the suspended particles be greater than the attractive dispersion force. Repulsion results from electrostatic surface charges caused by adsorption of electrolyte ions in the liquid medium. In the old but not yet retired Derjaguin-Landau-Verwey-Overbeek (DLVO) theory the repulsive potential energy between two colloidal particles is calculated as the energy required to remove the ions, and the repulsive force is compared to the dispersion force between the particles. If the dispersion force is larger than the repulsive force, the colloid is unstable.

In the DLVO theory it is assumed that the attractive (van der Waals) interaction varies as the inverse sixth power of the distance  $r$  between molecules, and that this force is pairwise additive. However, based on their experimental work, Verwey and Overbeek [5] wrote that, “In applying the theory of the attractive forces ... we met with some difficulties ... because the London theory as such is not relativistically invariant, and by working out this idea we found that ... a relativistic correction may well become important.” According to them the finite speed of light should cause the intermolecular van der Waals interaction to fall off more rapidly with separation  $r$  than  $r^{-6}$ . Following the suggestion of Verwey and Overbeek, their colleagues at the Philips Laboratory in Eindhoven, Casimir and Polder [6], revisited the London calculation to include retardation, or in other words to include  $\exp(i\mathbf{k} \cdot \mathbf{r})$  in the matrix elements appearing in perturbation theory. Their calculation showed that, for intermolecular separations larger than about 137 Bohr radii typically, the interaction energy between two identical molecules with static (zero-frequency) polarizability  $\alpha$  is

$$V(r) = -\frac{23\hbar c}{4\pi r^7} \alpha^2, \quad (1.3)$$

as opposed to (1.2). It turned out, according to Verwey and Overbeek [5], that this result was “in fair accord” with what they concluded from their experiments.

Another feature of the van der Waals dispersion force is that it is *nonadditive*, as was noted early on by Langmuir [7]. This nonadditivity emerges clearly in the many-body theory summarized below, but it does not appear to have played any significant role in the original DLVO theory.

The simple form of (1.3) led Casimir, after a suggestion by Niels Bohr, to interpret it in terms of zero-point energy: “[Bohr] mumbled something about zero-point energy. That was all, but it put me on a new track ...” [8]. This in turn led him to derive the force (1) as another example of an effect attributable to zero-point energy.

## 1.2 Zero-Point Energy

The concept of zero-point energy seems to have first appeared in Planck’s “second theory” of blackbody radiation, and soon after it played a role in some of Einstein’s work [9]. In a paper by Einstein and Stern in 1913 it was noted that, without zero-point energy, the average energy of a harmonic oscillator of frequency  $\omega$  is

$$E = \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1} \cong k_B T - \frac{1}{2}\hbar\omega \quad (1.4)$$

if the thermal equilibrium temperature  $T$  satisfies  $k_B T \gg \hbar\omega$ . To satisfy the equipartition theorem to first order in  $\hbar$  in this classical limit we must include the

zero-point energy  $\frac{1}{2}\hbar\omega$ . From such considerations Einstein and Stern concluded that “the existence of zero-point energy [ $\frac{1}{2}\hbar\omega$ ] is probable”. A bit later, however, Einstein wrote to Ehrenfest that zero-point energy is “dead as a doornail” [10].

Experimental evidence for zero-point energy was reported by Mulliken in 1924 [11]. Consider an absorptive vibronic transition in a diatomic molecule A in which the vibrational quantum numbers of the upper and lower states of the transition are  $v'$  and  $v''$ , respectively. The transition frequency is approximately

$$v_A(v', v'') = v_e + \omega'_e \left( v' + \frac{1}{2} \right) - \omega''_e \left( v'' + \frac{1}{2} \right) \quad (1.5)$$

if anharmonic corrections are small. Here  $v_e$  is the electronic transition frequency and  $\omega'_e$  and  $\omega''_e$  are the vibrational frequencies of the two electronic states. The zero-point vibrational energies of the upper and lower states are then  $\frac{1}{2}\hbar\omega'_e$  and  $\frac{1}{2}\hbar\omega''_e$ , respectively. Now consider a second diatomic molecule B that differs only isotopically from A; for this molecule, similarly, the transition frequency is

$$v_B(v', v'') = v_e + \rho\omega'_e \left( v' + \frac{1}{2} \right) - \rho\omega''_e \left( v'' + \frac{1}{2} \right), \quad (1.6)$$

where  $\rho = \sqrt{m_A/m_B}$  and  $m_A, m_B$  are the reduced masses of the two molecules. For the 0–0 bands,

$$v_B(0, 0) - v_A(0, 0) = \frac{1}{2}(\rho - 1)(\omega'_e - \omega''_e), \quad (1.7)$$

which is nonzero only because of the zero-point vibrational energies of the two molecules. Such an isotopic displacement was observed by Mulliken in the 0-0 bands for  $B^{10}O^{16}$  and  $B^{11}O^{16}$ . He concluded that “it is then probable that the minimum vibrational energy of BO (and doubtless other) molecules is 1/2 quantum”.

Of course one can cite other evidence for zero-point energy. For example, because of their small masses,  $He^3$  and  $He^4$  do not solidify at small pressures as  $T \rightarrow 0$  because their zero-point motion prevents crystallization. Many years ago Debye noted that the zero-point translational energy of the atoms of a crystal lattice causes a reduction in the intensity of radiation in X-ray diffraction even as the temperature approaches absolute zero. In more recent years the zero-point momentum distribution of atoms in Bose-Einstein condensates has been found to have the smallest width consistent with the Heisenberg uncertainty relation [12].

One of the most frequently cited implications of zero-point electromagnetic energy is the Lamb shift, or more specifically the dominant contribution to the Lamb shift in atomic hydrogen. An argument originally due to Feynman goes as follows [13]. Imagine we have a box of volume  $V$  containing  $N$  identical atoms per unit volume. The frequencies of the allowed field modes in the box are changed from their vacuum values  $\omega$  to  $\omega/n(\omega)$ , where  $n(\omega) \cong 1 + 2\pi N\alpha(\omega)$  is the refractive index of the (dilute) gas. The change in the zero-point field energy in the box due to the presence of the gas is therefore

$$\Delta E = \sum_{\mathbf{k}, \lambda} \left[ \frac{1}{2} \frac{\hbar \omega_{\mathbf{k}}}{n(\omega_{\mathbf{k}})} - \frac{1}{2} \hbar \omega_{\mathbf{k}} \right] \cong - \sum_{\mathbf{k}, \lambda} [n(\omega_{\mathbf{k}}) - 1] \frac{1}{2} \hbar \omega_{\mathbf{k}} = -\pi \hbar N \sum_{\mathbf{k}, \lambda} \omega_{\mathbf{k}} \alpha(\omega_{\mathbf{k}}), \quad (1.8)$$

where the  $\mathbf{k}$ 's and  $\lambda$ 's are the mode wave vectors and polarization labels, respectively. For a large box we can replace the discrete summation over modes by an integration:  $\sum_{\mathbf{k}, \lambda} \rightarrow (V/8\pi^3) \sum_{\lambda} \int d^3k$ . Then, in the limit of a single atom ( $NV \rightarrow 1$ ),

$$\Delta E = -\frac{\hbar}{\pi c^3} \int d\omega \omega^3 \alpha(\omega). \quad (1.9)$$

Subtracting the free-electron energy, and introducing a high-frequency cutoff  $mc^2/\hbar$  in this nonrelativistic approach, one obtains Bethe's approximation to the Lamb shift of a one-electron atom in a state with polarizability  $\alpha(\omega)$  [14]. For the  $2s_{1/2} - 2p_{1/2}$  Lamb shift in hydrogen, Bethe computed 1040 MHz, in good agreement with the measured shift of about 1058 MHz. Thus the largest part of this Lamb shift is attributable to the change in the zero-point field energy due to the mere presence of the atom.

We can think of this a little differently by first recalling that the energy involved in inducing an electric dipole moment  $\mathbf{d} = \alpha \mathbf{E}$  in an electric field  $\mathbf{E}$  is

$$W = -(1/2)\alpha \mathbf{E}^2. \quad (1.10)$$

For an atom in a state with polarizability  $\alpha(\omega)$  in a field of frequency  $\omega$ , this energy is just the quadratic ac Stark shift of the state. Now for an atom in vacuum there is a continuum of field frequencies, and we obtain the energy shift due to the zero-point field by integrating over all frequencies. In the integrand in this case we set  $(1/4\pi)\mathbf{E}^2$  equal to  $\rho_0(\omega)d\omega$ , where  $\rho_0(\omega)d\omega$  is the zero-point field energy per unit volume in the frequency interval  $[\omega, \omega + d\omega]$ . Then

$$W = -\frac{1}{2} \int_0^{\infty} \alpha(\omega) 4\pi \rho_0(\omega) d\omega, \quad (1.11)$$

which is the same as (1.9), since

$$\rho_0(\omega) = \frac{\omega^2}{\pi^2 c^3} \left( \frac{1}{2} \hbar \omega \right). \quad (1.12)$$

In other words, the Lamb shift can be regarded as a Stark shift caused by the vacuum electric field.

The same sort of argument can be used to obtain the van der Waals dispersion interaction between two polarizable particles. Thus, for atom A, there is a contribution of the form (1.10) from each frequency  $\omega$  of the field, and this contribution depends on the polarizability  $\alpha_A(\omega)$  of atom A. The field  $\mathbf{E}(\mathbf{r}_A, t)$  acting on

atom A is the vacuum field  $\mathbf{E}_0(\mathbf{r}_A, t)$  at  $\mathbf{r}_A$  plus the field  $\mathbf{E}_B(\mathbf{r}_A, t)$  at  $\mathbf{r}_A$  due to atom B. The latter field is just the field from the electric dipole induced in B by the vacuum field at  $\mathbf{r}_B$ , and this brings in the polarizability  $\alpha_B(\omega)$  of atom B. We thus obtain for atom A (and likewise for atom B) an energy that depends in part on  $r = |\mathbf{r}_A - \mathbf{r}_B|$ ; this  $r$ -dependent interaction energy has the general form

$$V(r) \propto -\frac{\hbar}{c^3 r^3} \int_0^\infty d\omega \omega^3 \alpha_A(\omega) \alpha_B(\omega) G(\omega r/c). \quad (1.13)$$

The detailed functional form of  $G(\omega r/c)$ , which is an oscillatory function of  $\omega r/c$ , need not concern us, as (1.13) as it stands is sufficient to determine the form of the interaction at large separations: for large  $r$  we expect substantial cancellations due to the oscillatory nature of  $G(\omega r/c)$ , and on this basis we expect to obtain the correct  $r$ -dependence of  $V(r)$  by cutting off the upper limit of integration at a frequency  $\sim c/r$ . Then [15]

$$V(r) \propto -\frac{\hbar}{c^3 r^3} \alpha^2 \int_0^{c/r} d\omega \omega^3 G(\omega r/c) = -\frac{\hbar c}{r^7} \alpha^2 \int_0^1 dx x^3 G(x), \quad (1.14)$$

where we have assumed the two atoms to be identical and replaced their polarizability  $\alpha(\omega)$  by the static polarizability  $\alpha$  under the assumption that  $c/r$  is much smaller than the frequency of any transition that makes a significant contribution to  $\alpha(\omega)$ . Thus we obtain in this way the correct  $r$  dependence of the retarded van der Waals interaction.

We can derive the *exact* form of the van der Waals interaction *for all*  $r$  by this approach based on zero-point field energy. Neither molecule in the dispersion interaction has a permanent dipole moment; each has a fluctuating dipole moment *induced by the vacuum field* at its position, and from this perspective there is a nonvanishing force because the vacuum field correlation function  $\langle E_i(\mathbf{r}_A, t) E_j(\mathbf{r}_B, t) \rangle \neq 0$  [14]. That is, the dipole moments induced by the fluctuating zero-point electric field are correlated over finite differences, leading to a nonvanishing expectation value of the intermolecular interaction energy. In a similar fashion we can obtain, for instance, the van der Waals interaction for magnetically polarizable particles [16].

Another effect of retardation considered by Casimir and Polder [6] is the potential energy of an atom in the vicinity of a perfectly conducting wall. For short distances  $d$  between the atom and the wall the potential  $V(d)$  may be deduced from the electric dipole-dipole interaction between the atom and its image in the wall, and obviously varies as  $1/d^3$ . At large distances, however, retardation becomes important and, as in the interatomic van der Waals interaction, the interaction is reduced by a factor  $1/(\text{distance})$ , i.e.,  $V(d) \propto 1/d^4$ .

Here again the simple formula (1.10) provides a way to a simple derivation of this result of Casimir and Polder. The Stark shift (1.10) for an atom in a state with polarizability  $\alpha(\omega)$  in this example becomes

$$W = -\frac{1}{2} \sum_{\mathbf{k}, \lambda} \alpha(\omega) \mathbf{E}_{\mathbf{k}, \lambda}^2(\mathbf{x}_A), \quad (1.15)$$

where the summation is over all modes,  $\mathbf{k}$  and  $\lambda$  again denoting wave vectors and polarizations, and  $\mathbf{E}_{\mathbf{k}, \lambda}(\mathbf{x}_A)$  is the zero-point electric field from mode  $(\mathbf{k}, \lambda)$  at the position  $\mathbf{x}_A$  of the atom. In the half-space in which the atom is located the mode functions are determined, of course, by Maxwell's equations and the boundary conditions on the fields. Consider first a rectangular parallelepiped with sides of length  $L_x = L_y = L$  and  $L_z$ , its surfaces being assumed to be perfectly conducting. The zero-point electric field  $\mathbf{E}$  inside the parallelepiped can be expanded in terms of a complete set of mode functions such that the the boundary conditions are satisfied and

$$\frac{1}{4\pi} \int d^3r \mathbf{E}^2(\mathbf{r}) = \sum_{\mathbf{k}, \lambda} \frac{1}{2} \hbar \omega_k. \quad (1.16)$$

The Cartesian components of  $\mathbf{E}$  for each mode  $(\mathbf{k}, \mathbf{e}_{\mathbf{k}, \lambda})$  are

$$\begin{aligned} E_x(\mathbf{r}) &= \left( \frac{16\pi\hbar\omega}{V} \right)^{1/2} e_x \cos(k_x x) \sin(k_y y) \sin(k_z z), \\ E_y(\mathbf{r}) &= \left( \frac{16\pi\hbar\omega}{V} \right)^{1/2} e_y \sin(k_x x) \cos(k_y y) \sin(k_z z), \\ E_z(\mathbf{r}) &= \left( \frac{16\pi\hbar\omega}{V} \right)^{1/2} e_z \sin(k_x x) \sin(k_y y) \cos(k_z z), \end{aligned} \quad (1.17)$$

where  $e_x^2 + e_y^2 + e_z^2 = 1$ , the volume  $V = L^2 L_z$ , and

$$k_x = \frac{\ell\pi}{L}, \quad k_y = \frac{m\pi}{L}, \quad k_z = \frac{n\pi}{L_z}. \quad (1.18)$$

All positive integers and zero are allowed for  $\ell, m$ , and  $n$ . As in the derivation of (1.14) we replace  $\alpha(\omega)$  by the static polarizability  $\alpha = \alpha(0)$  in (1.15) and write, for an atom at  $(L/2, L/2, d)$ ,

$$\begin{aligned} W(d) &= -\frac{1}{2} \alpha \sum_{\mathbf{k}} \frac{16\pi\omega_k}{V} \left[ e_x^2 \cos^2\left(\frac{1}{2}k_x L\right) \sin^2\left(\frac{1}{2}k_y L\right) \sin^2(k_z d) \right. \\ &\quad + e_y^2 \sin^2\left(\frac{1}{2}k_x L\right) \cos^2\left(\frac{1}{2}k_y L\right) \sin^2\left(\frac{1}{2}k_z d\right) \\ &\quad \left. + e_x^2 \sin^2\left(\frac{1}{2}k_x L\right) \sin^2\left(\frac{1}{2}k_y L\right) \cos^2\left(\frac{1}{2}k_z d\right) \right]. \end{aligned} \quad (1.19)$$

The squares of the sines and cosines involving  $k_x$  and  $k_y$  are rapidly varying and can be replaced by their average value,  $1/2$ , so that (1.19) is replaced by

$$W(d) = -\left(\frac{2\pi\hbar\alpha}{V}\right) \sum_{\mathbf{k}} \omega_k \left[ (e_x^2 + e_y^2) \sin^2(k_z d) + e_z^2 \cos^2(k_z d) \right]. \quad (1.20)$$

In the limit  $d \rightarrow \infty$  in which the atom is infinitely far away from the wall at  $z = 0$  we can also replace  $\sin^2(k_z d)$  and  $\cos^2(k_z d)$  by their averages:

$$W(\infty) = -\left(\frac{2\pi\hbar\alpha}{V}\right) \sum_{\mathbf{k}} \omega_k \left[ \frac{1}{2} (e_x^2 + e_y^2 + e_z^2) \right] = -\left(\frac{2\pi\hbar\alpha}{V}\right) \sum_{\mathbf{k}} \frac{1}{2} \omega_k. \quad (1.21)$$

We define the interaction energy as

$$\begin{aligned} V(d) &= W(d) - W(\infty) = -\left(\frac{2\pi\hbar\alpha}{V}\right) \sum_{\mathbf{k}} \omega_k [e_x^2 + e_y^2 - e_z^2] \left[ \sin^2(k_z d) - \frac{1}{2} \right] \\ &= \left(\frac{\pi\hbar\alpha}{V}\right) \sum_{\mathbf{k}} \omega_k \cos(2k_z d) \times (2k_z^2/k^2), \end{aligned} \quad (1.22)$$

where we have used the fact that  $\mathbf{e}_{\mathbf{k}\lambda}^2 = 1$  and  $\mathbf{k} \cdot \mathbf{e}_{\mathbf{k}\lambda} = 0$  for each mode.  $V(d)$  is easily evaluated when we use the fact that there is a continuum of allowed  $\mathbf{k}$  vectors and that  $\omega_k = kc$ :

$$\begin{aligned} V(d) &= \left(\frac{2\pi\hbar\alpha}{V}\right) \frac{V}{8\pi^3} \int d^3k \frac{k_z^2}{k^2} \cos(2k_z d) \\ &= \left(\frac{\alpha\hbar c}{2\pi}\right) \int_0^\infty dk k^3 \int_0^{2\pi} d\theta \sin\theta \cos^2\theta \cos(2kd \cos\theta) \\ &= -\frac{3\alpha\hbar c}{8\pi d^4}, \end{aligned} \quad (1.23)$$

which is exactly the result of Casimir and Polder [6].

We cannot invoke directly such arguments based on (1.10) in the case of the Casimir force (1.1), but, as Casimir showed, we can obtain the force by calculating the change in the zero-point field energy when the plates are separated by a distance  $d$  compared to when they are infinitely far apart. Since Casimir's calculation is reproduced in various ways in many other places, we will here present only a simple, heuristic derivation.

Recall that there are  $(\omega^2/\pi^2 c^3)d\omega$  modes in the (angular) frequency interval  $[\omega, \omega + d\omega]$ , and that each mode has a zero-point energy  $\hbar\omega/2$ . Between the plates the components of the mode vectors perpendicular to the plates are restricted to values  $< \pi/d$ . There is no such restriction on modes propagating in the two directions parallel to the plates, so we might guess that, to obtain approximately the zero-point field energy in a volume  $Ad$  between the plates, we can use the *free-space* energy density with a lower bound  $\beta\pi c/d$ , where  $\beta \sim 1/3$ :



$$E = Ad \int_{\pi c\beta/d}^{\infty} d\omega \left[ \frac{1}{2} \hbar\omega \times \frac{\omega^2}{\pi^2 c^3} \right] = Ad \int_0^{\infty} d\omega \frac{\hbar\omega^3}{2\pi^2 c^3} - Ad \int_0^{\pi c\beta/d} d\omega \frac{\hbar\omega^3}{2\pi^2 c^3}. \quad (1.24)$$

Ignoring the first term in the second equality, which is a bulk-volume contribution and would imply an infinite force, independent of  $d$ , we define the potential energy

$$V(d) = -Ad \int_0^{\pi c\beta/d} d\omega \frac{\hbar\omega^3}{2\pi^2 c^3} = -Ad \frac{\hbar}{8\pi^2 c^3} \left( \frac{\pi c\beta}{d} \right)^4. \quad (1.25)$$

As noted,  $\beta$  should be around  $1/3$ ; let us take  $\beta = 0.325 = (1/90)^{1/4}$ . Then

$$V(d) = -A \frac{\pi^2 \hbar c}{720 d^3}, \quad (1.26)$$

or  $F(d) = -V'(d)/A = -\pi^2 \hbar c / 240 d^4$ , which is Casimir's result (1.1). Of course a more serious derivation should also take account of the effects of zero-point fields in the regions outside the two plates, but in the present example we can ignore them [17].

### 1.3 The Lifshitz Theory and Its Generalizations

In his generalization of Casimir's theory for perfect conductors at zero temperature, Lifshitz [2] considered two dielectric half-spaces separated by a vacuum region of width  $d$  and allowed for finite (equilibrium) temperatures. He calculated the force between the dielectrics using the macroscopic Maxwell equations and the stress tensor in the vacuum region, assuming a noise polarization consistent with the fluctuation-dissipation theorem. Lifshitz's approach employs stochastic fields, but is equivalent to a formulation based on operator-valued fields [18]. The (positive-frequency) Fourier transform  $\hat{\mathbf{E}}(\mathbf{r}, \omega)$  of the electric field operator  $\hat{\mathbf{E}}(\mathbf{r}, t)$  satisfies

$$-\nabla \times \nabla \times \hat{\mathbf{E}}(\mathbf{r}, \omega) + \frac{\omega^2}{c^2} \varepsilon(\mathbf{r}, \omega) \hat{\mathbf{E}}(\mathbf{r}, \omega) = -\frac{\omega^2}{c^2} \hat{\mathbf{K}}(\mathbf{r}, \omega), \quad (1.27)$$

with the noise polarization operator  $\hat{\mathbf{K}}(\mathbf{r}, \omega)$  having the properties

$$\langle \hat{\mathbf{K}}_i^\dagger(\mathbf{r}, \omega) \hat{\mathbf{K}}_j(\mathbf{r}', \omega') \rangle = 4\hbar \varepsilon_I(\omega) \delta_{ij} \delta(\omega - \omega') \delta^3(\mathbf{r} - \mathbf{r}') \frac{1}{e^{\hbar\omega/k_B T} - 1}, \quad (1.28)$$

$$\langle \hat{\mathbf{K}}_i(\mathbf{r}, \omega) \hat{\mathbf{K}}_j^\dagger(\mathbf{r}', \omega') \rangle = 4\hbar \varepsilon_I(\omega) \delta_{ij} \delta(\omega - \omega') \delta^3(\mathbf{r} - \mathbf{r}') \left[ \frac{1}{e^{\hbar\omega/k_B T} - 1} + 1 \right], \quad (1.29)$$

where  $\varepsilon_I(\omega)$  is the imaginary part of the permittivity  $\varepsilon(\omega)$ . For simplicity we restrict ourselves here to zero temperature, so that expectation values will refer to the ground state of the matter-field system rather than a finite-temperature thermal equilibrium state.

We first outline a derivation based on a stress tensor that leads to a rather general expression (1.39) from which Lifshitz's formula for the force is obtained. Recall first that in classical electromagnetic theory it follows from the macroscopic Maxwell equations that the force density in a dielectric medium in which there are electric and magnetic fields, but no free charges or currents, has components  $f_i = \partial_j T_{ij}$ , where the stress tensor

$$T_{ij} = \frac{1}{4\pi} \left[ E_i D_j + H_i H_j - \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{H}) \delta_{ij} \right]. \quad (1.30)$$

Then

$$f_i = \frac{1}{8\pi} [(\partial_i E_j) D_j - E_j (\partial_i D_j)] \quad (1.31)$$

when it is assumed that  $(\partial/\partial t)(\mathbf{D} \times \mathbf{H})/4\pi c$ , the rate of change of the Minkowski expression for the momentum density of the field, can be taken to be zero, as is appropriate under equilibrium conditions. We are also assuming isotropic media, in which case the Minkowski stress tensor (1.30) is symmetric.

When the field is quantized we replace  $E_j$  and  $D_j$  in (1.31) by operators, symmetrize, and take expectation values:

$$f_i = \frac{1}{8\pi} \text{Re} \left[ \langle (\partial_i \hat{E}_j) \hat{D}_j \rangle - \langle \hat{E}_j (\partial_i \hat{D}_j) \rangle \right]. \quad (1.32)$$

In terms of the Fourier transforms of the electric and displacement fields,

$$f_i(\mathbf{r}) = \frac{1}{8\pi} \text{Re} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \left\{ \langle [\partial_i \hat{E}_j(\mathbf{r}, \omega)] \hat{D}_j(\mathbf{r}, \omega') \rangle - \langle \hat{E}_j(\mathbf{r}, \omega) [\partial_i \hat{D}_j(\mathbf{r}, \omega')] \rangle \right\}. \quad (1.33)$$

Since  $\hat{\mathbf{D}}(\mathbf{r}, \omega) = \epsilon(\mathbf{r}, \omega) \hat{\mathbf{E}}(\mathbf{r}, \omega) + \hat{\mathbf{K}}(\mathbf{r}, \omega)$ ,

$$\langle \hat{E}_j(\mathbf{r}, \omega) \hat{D}_j(\mathbf{r}', \omega') \rangle = \epsilon(\mathbf{r}', \omega') \langle \hat{E}_j(\mathbf{r}, \omega) \hat{E}_j(\mathbf{r}', \omega') \rangle + \langle \hat{E}_j(\mathbf{r}, \omega) \hat{K}_j(\mathbf{r}', \omega') \rangle. \quad (1.34)$$

In the second term on the right we use

$$\hat{E}_j(\mathbf{r}, \omega) = \frac{1}{4\pi} \int d^3 r'' G_{ji}(\mathbf{r}, \mathbf{r}'', \omega) \hat{K}_i(\mathbf{r}'', \omega), \quad (1.35)$$

where  $G$  is the dyadic Green function satisfying

$$-\nabla \times \nabla \times G(\mathbf{r}, \mathbf{r}', \omega) + \frac{\omega^2}{c^2} \varepsilon(\mathbf{r}, \omega) G(\mathbf{r}, \mathbf{r}', \omega) = -4\pi \frac{\omega^2}{c^2} \delta^3(\mathbf{r} - \mathbf{r}'), \quad (1.36)$$

while in the first term [19],

$$\langle \hat{E}_j(\mathbf{r}, \omega) \hat{E}_j(\mathbf{r}', \omega') \rangle = \frac{\hbar}{\pi} \text{Im} G_{jj}(\mathbf{r}, \mathbf{r}', \omega) \delta(\omega - \omega'). \quad (1.37)$$

It follows from the (zero-temperature) fluctuation-dissipation relations (1.28) and (1.29) that

$$\text{Re} \langle \hat{E}_j(\mathbf{r}, \omega) \hat{D}_j(\mathbf{r}', \omega') \rangle = \frac{\hbar}{\pi} \text{Im} [\varepsilon(\mathbf{r}', \omega) G_{jj}(\mathbf{r}, \mathbf{r}', \omega)] \delta(\omega - \omega'), \quad (1.38)$$

and, from (1.33),

$$f_i(\mathbf{r}) = -\frac{\hbar}{8\pi^2} \text{Im} \int_0^\infty d\omega [\partial_i \varepsilon(\mathbf{r}, \omega)] G_{jj}(\mathbf{r}, \mathbf{r}, \omega). \quad (1.39)$$

Equation (1.39) gives the force density due to (non-additive) intermolecular van der Waals dispersion interactions. Its evaluation requires, however, a calculation of the (classical) dyadic Green function, which is generally very hard to do analytically except in a few highly symmetric configurations. In the case of two dielectric half-spaces, for example, (1.39) has been used to derive the Lifshitz expression for the force per unit area between two dielectric half-spaces, and to generalize the original Lifshitz formula to allow for a third dielectric material between the half-spaces [20–23].

Dzyaloshinskii et al. [20–22] obtained (1.39) in a diagrammatic approach leading to a free energy  $\mathcal{E}$  having the form of a summation over all  $n$ -body van der Waals interactions:

$$\mathcal{E} = -\frac{\hbar}{2\pi} \text{Im} \int_0^\infty d\omega \text{Tr} [\alpha G^0 + \frac{1}{2} \alpha^2 G^0 G^0 + \frac{1}{3} \alpha^3 G^0 G^0 G^0 + \dots]. \quad (1.40)$$

Here  $G^0$  is the free-space Green function satisfying (1.36) with  $\varepsilon(\omega) = 1$  for all frequencies  $\omega$ , and  $\alpha(\omega)$  is again the complex, frequency-dependent polarizability of each particle. The first term in brackets is a single-particle self-energy, while the other terms correspond successively to two-body, three-body, etc. van der Waals interactions. The second term, for example, written out explicitly as

$$\text{Tr} \left[ \frac{1}{2} \alpha^2 G^0 G^0 \right] = \sum_{m,n=1}^{\mathcal{N}} \alpha^2 G_{ij}^0(\mathbf{r}_n, \mathbf{r}_m, \omega) G_{ji}^0(\mathbf{r}_m, \mathbf{r}_n, \omega), \quad (1.41)$$

gives, excluding terms with  $m = n$ , the sum of pairwise van der Waals interaction energies of  $\mathcal{N}$  atoms for arbitrary interparticle separations  $|\mathbf{r}_n - \mathbf{r}_m|$ . The third

term in brackets in (1.40), similarly, can be shown to yield the three-body van der Waals interaction obtained by Axilrod and Teller [24]. In the model in which the atoms form a continuous medium, equation (1.41), for instance, is replaced by

$$\int d^3r \int d^3r' N(\mathbf{r})N(\mathbf{r}')\alpha^2 G_{ij}^0(\mathbf{r}, \mathbf{r}', \omega)G_{ji}^0(\mathbf{r}', \mathbf{r}, \omega) \equiv \text{Tr} \left[ \frac{\varepsilon - 1}{4\pi} \right]^2 [G^0]^2,$$

where  $N(\mathbf{r})$  is the number of atoms per unit volume at  $\mathbf{r}$  and the formula  $\varepsilon(\omega) = 1 + 4\pi N\alpha(\omega)$  has been used to relate the permittivity and the polarizability. Similarly (1.40) is replaced by

$$\mathcal{E} = -\frac{\hbar}{2\pi} \text{Im} \int_0^\infty d\omega \text{Tr} \sum_{n=1}^\infty \frac{1}{n} \left[ \frac{\varepsilon - 1}{4\pi} \right]^n [G^0]^n \quad (1.42)$$

in the continuum approximation.

Since the free-space Green dyadic  $G^0$  is independent of any properties of the medium, a variation  $\delta\mathcal{E}$  of  $\mathcal{E}$  can depend only on a variation  $\delta\varepsilon$  of  $\varepsilon$ :

$$\delta\mathcal{E} = -\frac{\hbar}{8\pi^2} \text{Im} \int_0^\infty d\omega \delta\varepsilon \text{Tr} \sum_{n=0}^\infty \left[ \frac{\varepsilon - 1}{4\pi} \right]^n [G^0]^{n+1}. \quad (1.43)$$

From (1.36) we obtain the identity

$$G = G^0 + G^0 \left[ \frac{\varepsilon - 1}{4\pi} \right] G, \quad (1.44)$$

which allows us to rewrite (1.43) as

$$\delta\mathcal{E} = -\frac{\hbar}{8\pi^2} \text{Im} \int_0^\infty d\omega \int d^3r \delta\varepsilon(\mathbf{r}, \omega) G_{ii}(\mathbf{r}, \mathbf{r}, \omega). \quad (1.45)$$

Now an infinitesimal local transport  $\mathbf{u}(\mathbf{r})$  of the medium implies a variation in  $\varepsilon$  at  $\mathbf{r}$  such that  $\varepsilon(\mathbf{r}, \omega) + \delta\varepsilon(\mathbf{r}, \omega) = \varepsilon(\mathbf{r} - \mathbf{u}, \omega)$ , or  $\delta\varepsilon(\mathbf{r}, \omega) = -\nabla\varepsilon \cdot \mathbf{u}$ . Therefore

$$\delta\mathcal{E} = -\frac{\hbar}{8\pi^2} \text{Im} \int_0^\infty d\omega [-\nabla\varepsilon \cdot \mathbf{u}] G_{ii}(\mathbf{r}, \mathbf{r}, \omega) = -\int d^3r \mathbf{f}(\mathbf{r}) \cdot \mathbf{u}, \quad (1.46)$$

where the Casimir force density  $\mathbf{f}(\mathbf{r})$  is defined by (1.39). The quantum-field-theoretic approach of Dzyaloshinskii et al. [20–22] thus establishes the connection between the non-additive, many-body van der Waals interactions and macroscopic quantum-electrodynamical approaches such as the stress-tensor formulation outlined above.

Another approach to the calculation of Casimir forces between dielectrics, based directly on considerations of zero-point energy, was taken van Kampen et al. [25]

and others in a rederivation of Lifshitz's formula for the force between two dielectric half-spaces. In this approach Maxwell's equations and the appropriate boundary conditions lead to a requirement that the allowed (nontrivial) modes satisfy  $F_\beta(\omega, d) = 0$ , where the  $F_\beta$  are meromorphic functions of frequency. The total zero-point energy associated with all the allowed modes, i.e.,  $\hbar/2$  times the sum of the zeros of the functions  $F_\beta$ , is then obtained from a generalization of the argument theorem for meromorphic functions, and from this one can obtain the Lifshitz formula.

A fundamental assumption made by Lifshitz is that the dielectric media are well described by continua characterized by dielectric permittivities. This allows the sum of all the many-body van der Waals forces to be obtained via the *macroscopic* Maxwell equations [26]. This assumption for material media is made, usually implicitly, in practically all theories of Casimir forces, and it appears to be an excellent approximation under practical experimental circumstances. One of the most accurate tests of the Lifshitz formula for the force between dielectric half-spaces was conducted many years ago by Sabisky and Anderson [27]. Using measured data and an analytic fit for the dielectric constants, they compared the predictions of the Lifshitz formula for the variation of the vapor pressure with film thickness of liquid helium, and reported agreement to within a few per cent between theory and experiment.

The formula for the force  $F(d)$  per unit area obtained by Lifshitz [2] can be generalized not only to include a material medium between the half-spaces but also to allow all three media to be magnetodielectric:

$$F(d) = -\frac{\hbar}{2\pi^2} \int_0^\infty dk k \int_0^\infty d\xi K_3 \left( \left[ \frac{(\varepsilon_3 K_1 + \varepsilon_1 K_3)(\varepsilon_3 K_2 + \varepsilon_2 K_3)}{(\varepsilon_3 K_1 - \varepsilon_1 K_3)(\varepsilon_3 K_2 - \varepsilon_2 K_3)} e^{2K_3 d} - 1 \right]^{-1} \right. \\ \left. + \left[ \frac{(\mu_3 K_1 + \mu_1 K_3)(\mu_3 K_2 + \mu_2 K_3)}{(\mu_3 K_1 - \mu_1 K_3)(\mu_3 K_2 - \mu_2 K_3)} e^{2K_3 d} - 1 \right]^{-1} \right), \quad (1.47)$$

where  $\varepsilon_j$  and  $\mu_j$  are now respectively the electric permittivity and the magnetic permeability of medium  $j(=1, 2, 3)$  evaluated at imaginary frequencies:  $\varepsilon_j = \varepsilon_j(i\xi)$ ,  $\mu_j = \mu_j(i\xi)$ , and  $K_j^2 = k^2 + \varepsilon_j \mu_j \xi^2 / c^2$ . The terms depending explicitly on  $\varepsilon_j$  and  $\mu_j$  in this formula have the form of Fresnel reflection coefficients. In fact in a more computationally useful "scattering" approach, Casimir forces between objects in the case of more general geometries are expressed in terms of the scattering matrices [28, 29]. For example, for media in which only specular reflection occurs, the Casimir force per unit area in the special case of the Lifshitz geometry with vacuum between the half-spaces is given by

$$F(d) = -\frac{\hbar}{4\pi^3} \int_0^\infty d\xi \int d^2 k K_3 \text{Tr} \frac{\mathbf{R}_1 \cdot \mathbf{R}_2 e^{-2K_3 d}}{1 - \mathbf{R}_1 \cdot \mathbf{R}_2 e^{-2K_3 d}}. \quad (1.48)$$

$\mathbf{R}_1$  and  $\mathbf{R}_2$  are  $2 \times 2$  reflection matrices for *generally anisotropic* magnetodielectric media and  $K_3^2 = k^2 + \xi^2/c^2$ .

## 1.4 Overview of Experiments

The Casimir force is typically very small, even in the case of perfectly conducting plates. For two  $1 \times 1$  cm plates separated by  $1 \mu\text{m}$ , for example, equation (1.1) predicts a force of 0.013 dyne. This is comparable to the Coulomb force on the electron in the hydrogen atom, or to the gravitational force between two one-pound weights separated by half an inch, or to about 1/1000 of the weight of a housefly. Not surprisingly, therefore, it took quite a few years before it was unambiguously measured. Here we briefly review some of the earlier experiments. More detailed discussions of some of these experiments, as well as more recent work, can be found in the following chapters.

The earliest experiments measuring van der Waals forces between macroscopic objects were performed by Derjaguin et al. in the 1950s and earlier [30]. The alignment difficulties in experiments with parallel plates led this group to work instead with a spherical lens and a plate. It was shown that the force between a sphere of radius  $R$  and a flat surface a distance  $d$  away is approximately

$$\mathcal{F}(d) = 2\pi R u(d), \quad (1.49)$$

where  $u(d)$  is the interaction energy per unit area between two flat, parallel surfaces separated by  $d$ . This approximation, valid when  $d \ll R$ , has come to be known as the Derjaguin or “proximity force approximation (PFA)”, and although it is derived under the assumption of pairwise additive forces between local surface elements, it has been an important and surprisingly accurate tool in the comparison of measured forces with theories for perfectly flat, parallel surfaces [31]. Discussions and references to theoretical analyses of the range of validity of the PFA may be found in several of the following chapters. Measurements of the forces between a flat surface and spheres of different radii have demonstrated that the PFA is fairly accurate for values of  $d$  and  $d/R$  in the range of many experiments [32].

Experiments of Derjaguin et al. for separations of  $0.1\text{--}0.4 \mu\text{m}$  between quartz plates provided the stimulus for Lifshitz’s theory [30]. The comparison with the Lifshitz theory was complicated in part by incomplete information about  $\varepsilon(i\xi)$  for quartz over all its different absorption regions. On the basis of rough estimates, Lifshitz concluded that “the agreement between the theory and the experimental data is satisfactory” [2], although in his paper he provided few details. Forces measured much later were found to “fit well” with the Lifshitz theory in the retarded regime of large separations, but detailed comparisons of theory and experiments were again hampered by insufficient data for the permittivities required in the Lifshitz formula.

Sparnaay [33] reviewed experimental progress up to 1989, and some of the rapid progress, both theoretical and experimental, in more recent times has been the subject of reviews and special issues of journals that are extensively cited in the following chapters. In Sparnaay's own early experiments [34, 35] the force was inferred from the deflection of a spring attached via an aluminum rod to one of the metal plates. For plate separations of 2–10  $\mu\text{m}$ , Sparnaay measured attractive forces per unit area of magnitude between about 1 and  $4 \times 10^{-18}$   $\text{dyn cm}^2/d^4$ , compared with the  $1.3 \times 10^{-18}$   $\text{dyn cm}^2/d^4$  calculated from Casimir's formula (1.1). Prior to more recent work, Sparnaay's experiments were often cited as the first experimental verification of the Casimir force, but Sparnaay himself concluded only that his observed forces "do not contradict Casimir's theoretical prediction".

Experiments reported by Lamoreaux [36, 37] and Mohideen et al. [38–40] in 1997 and 1998 marked the beginning of a new era of much more precise measurements of Casimir forces. Lamoreaux performed Cavendish-type experiments employing torsion balances, whereas Mohideen's group pioneered the use of an atomic force microscope (AFM) system wherein the Casimir force is determined from the reflection of a laser beam off the AFM cantilever tip and its displacement as measured by photodiodes. The agreement of these experiments with theory appears to be on the order of perhaps several per cent. This great improvement in accuracy compared to earlier experiments stems in part from the availability of much better mechanical, surface characterization, and data acquisition tools. The measurements of Lamoreaux and Mohideen et al. also avoided the problem of stiction associated with previous balance mechanisms.

Another very significant step in the direction of improved accuracy of Casimir force measurements was reported by Chan et al. [41, 42] in 2001, who measured the Casimir force based on the shift in the frequency of a periodically driven MEMS torsional oscillator (MTO) or on the torque exerted on a plate by a metallic sphere. Two years later Decca et al [43] reported the first highly accurate measurement of the Casimir force between two dissimilar metals based on an MTO system.

De Man et al. [44, 45] have described AFM Casimir-force measurements that allow for continuous calibration together with compensation of residual electrostatic effects. Experiments measuring the force in air between a gold-coated sphere and a glass surface coated with either gold or a conductive oxide demonstrated that residual electrostatic effects could be effectively eliminated, while the Casimir force was substantially reduced, as expected, when the gold surface was replaced by the transparent oxide film. These authors have also introduced a "fiber-top cantilever" design that could allow Casimir force measurements under environmental conditions that would preclude measurements employing existing laboratory instrumentation.

Substantial progress has also been made in the measurement of the Casimir-Polder force. A particularly noteworthy experiment was that of Sukenik et al. [46] in which the opacity of a sodium beam passing through a gold-plated channel was

measured. Assuming that a sodium atom, hitting a gold surface as a consequence of the Casimir-Polder attractive force, sticks without bouncing, Sukenik et al. found excellent agreement between their data and the atom-surface interaction predicted by Casimir and Polder.

This brief overview of experimental progress is hardly meant to be exhaustive. In the remaining chapters there are  $\sim 10^3$  citations to the literature on Casimir effects, a significant portion of it addressing observed Casimir effects and related experimental issues.

## 1.5 Some Other Aspects of Casimir Forces

The following chapters provide detailed discussions of many of the most interesting theoretical as well as experimental aspects of Casimir effects. As expected from their relation to the ubiquitous van der Waals forces, Casimir effects are involved directly or indirectly in a wide variety of physical phenomena, and it is probably impossible to address all aspects of them in any detail in a single volume. Here we briefly mention a few related topics of current interest that are either not dealt with or only briefly alluded to in this book.

Most of the work on van der Waals–Casimir–Lifshitz forces between material bodies has dealt with perfect conductors or dielectric media. If we take  $\varepsilon_1 = \varepsilon_2 \rightarrow \infty$  and  $\varepsilon_3 = 1$  for all frequencies, we obtain from the Lifshitz formula (1.48) the attractive Casimir force (1.1) for two perfectly conducting parallel plates separated by a vacuum. For identical dielectric half-spaces separated by a vacuum, the force is likewise always attractive. If the dielectric media are different, however, the force can be *repulsive* under certain conditions, e.g., if  $\varepsilon_1 > \varepsilon_3 > \varepsilon_2$  over a sufficiently wide range of frequencies. Such a repulsion was predicted by Dzyaloshinskii et al. [20–22] and can be applied, for instance, to explain the tendency of liquid helium to climb the walls of its container: the helium vapor is repelled by the walls, and the helium liquid moves in to fill the space left by the vapor. The repulsive force resulting from regions of different permittivity appears to be well known among colloid scientists [47], and has recently been directly observed by Munday et al. [48], who measured a repulsive force between a gold sphere and a glass surface immersed in bromobenzene.

In recent years it has been recognized that stiction due to attractive Casimir forces should be taken into account in the design of nanomechanical devices. There has consequently been a growing interest in the control of Casimir forces and in particular in the possibility of realizing repulsive Casimir forces between vacuum-separated objects. This in turn has led to the consideration of metamaterials that might be engineered to have permittivities and permeabilities that would allow a degree of control of Casimir forces that would not be possible with naturally occurring materials. An indication of how magnetic properties of materials might be used to control Casimir forces can be seen by taking



$\varepsilon_1 = \mu_2 \rightarrow \infty$  and  $\varepsilon_3 = 1$  for all frequencies, in which case we obtain from (1.48) a repulsive force with a magnitude of 7/8 times the magnitude of the Casimir force, as first shown in a different way by Boyer [49]. In a similar vein it was shown by Feinberg and Sucher [50] that the van der Waals interaction between two atoms, one electrically polarizable and the other magnetically polarizable, is repulsive.

Computations based on the Lifshitz theory and formulas for permittivities and permeabilities of some existing metamaterials suggest, however, that repulsion will be difficult to realize [51]. In fact Rahi et al. [52] have shown under some rather general conditions of physical interest that Casimir interactions cannot produce a stable equilibrium for any collection of dielectric objects whenever all their permittivities are larger or smaller than the permittivity of the medium in which they are immersed. In the case of metamaterials their arguments indicate that repulsive forces between two objects might only be possible at short distances, in which case a metamaterial surface cannot be assumed to have continuous translational symmetry. Even in this case, however, a repulsion cannot result in a stable equilibrium.

Casimir forces are notoriously difficult to compute for arbitrary geometries. The best-known paradigm here is the Casimir force on a perfectly conducting spherical shell, which Casimir assumed would be attractive. As first shown by Boyer [53] after “a long nightmare of classical special function theory”, however, the force in this case is repulsive. In addition to the chapters in this volume addressing numerical approaches to Casimir-force computations, we call attention here to the work of Schaden [54], who has used a semiclassical approach to compute, among other things, the force on a spherical conducting spherical shell to an accuracy within 1% of the exact expression.

In various extensions of the Standard Model of elementary particles there appear non-gravitational, long-range forces between electrically neutral bodies. Experimental constraints on such forces can be determined by Casimir as well as gravitational experiments, and the rapid progress in high-precision Casimir experiments has contributed to an increased interest in this area. We refer the reader to the last chapter of the book by Bordag et al. [38–40] for a discussion of this topic and references to recent work.

Casimir effects are almost always interpreted in terms of zero-point field energy. It is, however, possible to interpret Casimir effects without reference to zero-point field energy. For instance, the Lifshitz formula can be derived from Schwinger’s source theory in which “the vacuum is regarded as truly a state with all physical properties equal to zero” [23]. In conventional quantum electrodynamics Casimir forces can also be obtained without explicit reference to vacuum field fluctuations, although of course such fluctuations are implicit in the theory [55]. In a different approach Jaffe [56] has also shown that Casimir effects can be calculated “without reference to zero-point energies”, and suggests that approaches based on zero-point energy “won out” because they are much simpler.

Finally we note that there are classical analogs of the Casimir force. A thermodynamic analog occurs in the case of two surfaces in a binary liquid mixture that is close to the critical point [57–59]. The analog of the electromagnetic field

fluctuations between plates in the Casimir effect is in this case the fluctuations in the concentrations in the liquid between the two surfaces. The force acting on the surfaces can be attractive or repulsive, depending on whether the adsorptive characteristics of the two surfaces are similar or dissimilar. Both attraction and repulsion in this “critical Casimir effect” have been observed in experiments measuring forces of the same order of magnitude as the forces observed in (quantum) Casimir-force experiments, and the measured forces confirmed theoretical predictions.

## 1.6 Brief Outline of this Book

The ordering of chapters in this volume is only by topic and not by degree of depth or specialization. The first five chapters are concerned primarily with the theory of surface-surface Casimir effects. Pitaevskii discusses the problems that were encountered in the generalization of the original Lifshitz theory to the case of forces within dielectric media, especially in connection with a general formulation of the stress tensor. He reviews how the desired generalization was finally achieved by many-body diagrammatic methods applicable under conditions of thermal equilibrium. The important role of mechanical equilibrium and pressure gradients in the theory of dielectric bodies separated by a fluid is emphasized, and a physical interpretation is given for the repulsive forces predicted by the generalized Lifshitz theory for dielectrics. The chapter by Milton focuses on the nature of the divergences incurred in calculations of Casimir self-energies, such as that for a conducting spherical shell, based on the model of a massless scalar field and delta-function potentials. He distinguishes between the global divergence associated with the total zero-point energy and the local divergences occurring near boundaries; a unique and finite self-energy is obtained after isolating the global divergence. In the case of parallel plates it is shown that both the finite Casimir self-energy and the divergent self-energies of the plates are consistent with the equivalence principle, and therefore that the divergent self-energies in particular can be absorbed into the masses of the plates. This suggests a general “renormalization” procedure for absorbing divergent self-energies into the properties of boundaries. The following three chapters describe techniques for the computation of Casimir effects for arbitrarily shaped objects. Lambrecht, Canaguier-Durand, Guéroult and Reynaud, and Rahi, Emig and Jaffe, focus on the recently developed scattering theory approach and provide valuable introductions to methods they have developed. Lambrecht et al describe computations for several different configurations and make interesting comparisons with the predictions of the PFA, while Rahi et al consider a different set of examples and also address in detail the constraints on stable equilibria presented in Reference [52]. Johnson reviews a wide variety of methods in classical computational electromagnetism that can be applied to the evaluation of various Casimir effects, and offers illuminating perspectives on the strengths and weaknesses of each approach.

The next four chapters describe various experimental aspects of surface-surface Casimir forces. Lamoreaux reviews some recent experimental progress and then discusses, among other things, some of the implicit approximations that have “wittingly or unwittingly” been adopted in all Casimir force experiments, including his own seminal experiments. These include the PFA and various assumptions about electrostatic calibrations and contact and patch potentials. He cautions against confusing precision and accuracy in Casimir force measurements. Capasso, Munday, and Chan discuss high-precision measurements in MEMS, including applications to nonlinear oscillators for position measurements at the nanoscale, and then describe their experiments demonstrating a reduction in the magnitude of Casimir forces by different effects. They also discuss their experiments on repulsive Casimir forces [41, 42] and possibilities for realizing quantum levitation and Casimir torques between birefringent materials. Decca, Aksyuk, and López discuss advantages of using MEMS for the measurement of Casimir forces; as mentioned earlier, these devices have been used to measure Casimir forces between metallic objects in vacuum to a very high degree of accuracy [43]. Decca et al also discuss electrostatic calibration and other matters involved in Casimir force experiments, and describe potential opto-mechanical experiments that might allow further improvements in the precision of Casimir force measurements. Van Zwol, Svetovoy, and Palasantzas address the requirement, for any comparison of theory and experiment, of having accurate values of optical permittivities; they emphasize that the permittivities of conducting films, which are needed for all frequencies in the Lifshitz theory, are not always reliably given by tabulated data, since they can vary significantly from sample to sample, depending on how the samples are prepared. They describe determinations of the complex permittivities of gold films over a wide range of frequencies. Van Zwol et al. also discuss the characterization of surface roughness by imaging methods, and how such images can be used to characterize surface roughness and to calculate the correction to the surface-surface Casimir force due to it. They discuss the importance of the “distance upon contact” between two rough surfaces and its importance in the determination of their absolute separation and therefore of the Casimir force.

Atom-surface and dynamical Casimir effects are the subject of the remaining three chapters. Intravaia, Henkel, and Antezza review the theory of the Casimir-Polder interaction and some recent developments relating, among other things, to non-equilibrium systems and experiments with ultracold atoms. They also discuss frictional effects on moving atoms in blackbody fields and near surfaces. De Kieviet, Jentschura, and Lach review the experimental status of the Casimir-Polder force. They discuss other effects that might play a role in future experiments and review their work on the quantum reflection of ground-state atoms from solid surfaces and on the atomic beam spin echo technique. They present new experimental data on the reflectivity of  $^3\text{He}$  atoms from a gold surface and compare the data with predictions of the Casimir-Polder and non-retarded van der Waals atom-surface interactions. Their approach makes it possible to resolve very

detailed features of the atom-surface potential. Finally Dalvit, Maia Neto, and Mazzitelli review the theory of dynamical Casimir effects as well as frictional forces associated with electromagnetic field fluctuations, and discuss possible experimental scenarios for the observation of such effects.

## References

1. Casimir, H.B.G.: On the attraction between two perfectly conducting plates. Proc. K. Ned. Akad. Wet. **51**, 793 (1948)
2. Lifshitz, E.M.: The theory of molecular attractive forces between solids. Sov. Phys. JETP **2**, 73 (1956)
3. London, F.: Theory and system of molecular forces. Z. Phys. **63**, 245 (1930)
4. See, for instance, V. A. Parsegian, *Van der Waals Forces: A Handbook for Biologists, Chemists, Engineers, and Physicists* (Cambridge University Press, N. Y., 2006).
5. Verwey, E.J.W., Overbeek, J.T.G.: *Theory of the Stability of Lyophobic Colloids*. Elsevier, Amsterdam (1948)
6. Casimir, H.B.G., Polder, D.: The influence of retardation on the London-van der Waals forces. Phys. Rev. **73**, 360 (1948)
7. Langmuir, I.: Role of attractive and repulsive forces in formation of tactoids, thixotropic gels, protein crystals and coacervates. J. Chem. Phys. **6**, 873 (1938)
8. Casimir, H.B.G.: Communication to P.W. Milonni, 12 March (1992)
9. Some of Einstein's work related to zero-point energy is reviewed in P.W. Milonni, *The Quantum Vacuum. An Introduction to Quantum Electrodynamics* (Academic, San Diego, 1994).
10. Gearhart, C.A.: 'Astonishing successes' and 'bitter disappointment': The specific heat of hydrogen in quantum theory. Arch. Hist. Exact Sci. **64**, 113 (2010)
11. Mulliken, R.S.: The band spectrum of boron monoxide. Nature **114**, 349 (1924)
12. Stenger, J., Inouye, S., Chikkatur, A.P., Stamper-Kurn, D.M., Pritchard, D.E., Ketterle, W.: Bragg spectroscopy of a Bose-Einstein condensate. Phys. Rev. Lett. **82**, 4569 (1999)
13. See, for instance, Milonni, P.W., Schaden, M., Spruch, L.: The Lamb shift of an atom in a dielectric medium. Phys. Rev. A **59**, 4259 (1999) and references therein
14. See, for example, Reference [9] and references therein
15. Spruch, L.L., Kelsey, E.J.: Vacuum fluctuation and retardation effects in long-range potentials. Phys. Rev. A. **18**, 845 (1978)
16. Feinberg, G., Sucher, J.J., Au, C.K.: The dispersion theory of dispersion forces. Phys. Rep. **180**, 83 (1978)
17. Itzykson, C., Zuber, J.-B.: *Quantum Field Theory*, pp. 141. McGraw-Hill, N.Y. (1980)
18. Rosa, F.S.S., Dalvit, D.A.R., Milonni, P.W.: Electromagnetic energy, absorption, and Casimir forces: Uniform dielectric media in thermal equilibrium. Phys. Rev. A. **81**, 033812 (2010)
19. Rosa, F.S.S., Dalvit, D.A.R., Milonni, P.W.: to be submitted for publication. Expressions equivalent to (37) but with different choices for the definition of the Green dyadic may be found, for instance, in T. Gruner and D.-G. Welsch, Green-function approach to the radiation-field quantization for homogeneous and inhomogeneous Kramers-Kronig dielectrics. Phys. Rev. A **53**, 1818 (1996) and M.S. Tomas, Casimir force in absorbing monolayers. Phys. Rev. A **66**, 052103 (2002)
20. Dzyaloshinskii, I.E., Pitaevskii, L.P.: Van der Waals forces in an inhomogeneous dielectric. Sov. Phys. JETP **9**, 1282 (1959)
21. Dzyaloshinskii, I.E., Lifshitz, E.M., Pitaevskii, L.P.: The general theory of van der Waals forces. Adv. Phys. **10**, 165 (1961)

22. See also A.A. Abrikosov, L.P. Gorkov, I.E. Dzyaloshinskii, *Methods of Quantum Field Theory in Statistical Physics* (Dover, N.Y., 1975)
23. Schwinger, J., DeRaad, L.L. Jr., Milton, K.A.: Casimir effect in dielectrics. *Ann. Phys. (N.Y.)* **115**, 1 (1978)
24. Axilrod, B.M., Teller, E.: Interaction of the van der Waals type between three atoms. *J.Chem. Phys.* **11**, 299 (1943)
25. van Kampen, N.G., Nijboer, B.R.A., Schram, K.: On the macroscopic theory of van der Waals forces. *Phys. Lett.* **26**, 307 (1968)
26. See, for instance, Reference [9], Sect. 8.3.
27. Sabisky, E.S., Anderson, C.H.: Verification of the Lifshitz theory of the van der Waals potential using liquid-helium films. *Phys. Rev. A* **7**, 790 (1973)
28. Kats, E.I.: Influence of nonlocality effects on van der Waals interaction. *Sov. Phys. JETP* **46**, 109 (1977)
29. For a detailed review of the scattering approach see, for instance, the chapters by A. Lambrecht et al and S.J. Rahi et al in this volume
30. Derjaguin, B.V., Rabinovich, Y.I., Churaev, N.V.: Direct measurement of molecular forces. *Nature* **272**, 313 (1978) and references therein to related work of Derjaguin et al
31. It should be noted that Bressi et al managed to measure with about 15% precision the Casimir force between (nearly) parallel metallic plates: G. Bressi, G. Carugno, R. Onofrio, and G. Ruoso, "Measurement of the Casimir force between parallel metallic surfaces," *Phys. Rev. Lett.* **88**, 041804 (2002)
32. Krause, D.E., Decca, R.S., López, D., Fischbach, E.: Experimental investigation of the Casimir force beyond the proximity-force approximation. *Phys. Rev. Lett.* **98**, 050403 (2007)
33. Sparnaay, M.J.: The historical background of the Casimir effect. In: Sarlemijn, A., Sparnaay, M.J. (eds) *Physics in the Making*, Elsevier, Amsterdam (1989)
34. Sparnaay, M.J.: Attractive forces between flat plates. *Nature* **180**, 334 (1957)
35. Sparnaay, M.J.: Measurements of attractive forces between flat plates. *Physica* **24**, 751 (1958)
36. Lamoreaux, S.: Demonstration of the Casimir force in the 0.6 to 6  $\mu\text{m}$  range. *Phys. Rev. Lett.* **78**, 5 (1997)
37. See also S. Lamoreaux, The Casimir force: background, experiments, and applications. *Rep. Prog. Phys.* **68**, 201 (2005) and "Casimir forces: Still surprising after 60 years," *Physics Today* (February, 2007), 40-45
38. Mohideen, U., Roy, A.: Precision measurement of the Casimir force from 0.1 to 0.9  $\mu\text{m}$ . *Phys. Rev. Lett.* **81**, 4549 (1998)
39. See also M. Bordag, U. Mohideen, V.M. Mostepanenko, "New developments in the Casimir effect," *Phys. Rep.* **353**, 1 (2001)
40. Bordag, M., Klimchitskaya, G.L., Mohideen, U., Mostepanenko, V.M.: *Advances in the Casimir Effect*. Oxford University Press, N.Y. (2009)
41. Chan, H.B., Aksyuk, V.A., Kleiman, R.N., Bishop, D.J., Capasso, F.: Quantum mechanical actuation of microelectromechanical systems by the Casimir force. *Science* **291**, 1941 (2001)
42. Chan, H.B., Aksyuk, V.A., Kleiman, R.N., Bishop, D.J., Capasso, F.: Nonlinear micromechanical Casimir oscillator. *Phys. Rev. Lett.* **87**, 211801 (2001)
43. Decca, R.S., López, D., Fischbach, E., Krause, D.E.: Measurement of the Casimir force between dissimilar metals. *Phys. Rev. Lett.* **91**, 504021 (2003)
44. de Man, S., Heeck, K., Wijnngaarden, R.J., Iannuzzi, D.: Halving the Casimir force with conductive oxides. *Phys. Rev. Lett.* **103**, 040402 (2009)
45. de Man, S., Heeck, K., Smith, K., Wijnngaarden, R.J., Iannuzzi, D.: Casimir force measurements in air: two birds with one stone. *Int. J. Mod. Phys. A* **25**, 2231 (2010)
46. Sukenik, C.I., Boshier, M.G., Cho, D., Sandoghar, V., Hinds, E.A.: Measurement of the Casimir-Polder force. *Phys. Rev. Lett.* **70**, 560 (1993)
47. See for instance A.A. Feiler, L. Bergstrom, M.W. Rutland, "Superlubricity using repulsive van der Waals forces," *Langmuir* **24**, 2274 (2008), and references therein.
48. Munday, J.N., Capasso, F., Parsegian, V.A.: Measured long-range Casimir-Lifshitz forces. *Nature* **457**, 170 (2009)

49. Boyer, T.H.: Van der Waals forces and zero-point energy for dielectric and permeable materials. *Phys. Rev. A* **9**, 2078 (1974)
50. Feinberg, G., Sucher, J.: General theory of the van der Waals interaction: A model-independent approach. *Phys. Rev. A* **2**, 2395 (1970)
51. Rosa, F.S.S., Dalvit, D.A.R., Milonni, P.W.: Casimir-Lifshitz theory and metamaterials. *Phys. Rev. Lett.* **100**, 183602 (2008); “Casimir interactions for anisotropic magnetodielectric metamaterials”, *Phys. Rev. A* **78**, 032117 (2008)
52. Rahi, S.J., Kardar, M., Emig, T.: Constraints on stable equilibria with fluctuation-induced (Casimir) forces. *Phys. Rev. Lett.* **105**, 070404 (2010)
53. Boyer, T.H.: Quantum zero-point energy and long-range forces. *Ann. Phys. (N.Y.)* **56**, 474 (1970)
54. Schaden, M.: Semiclassical estimates of electromagnetic Casimir self-energies of spherical and cylindrical metallic shells. *Phys. Rev. A* **82**, 022113 (2010)
55. See, for instance, Reference [9] and references therein.
56. Jaffe, R.L.: Casimir effect and the quantum vacuum. *Phys. Rev. D.* **72**, 021301(R) (2005)
57. Fisher, M.E., de Gennes, P.-G.: Wall phenomena in a critical binary mixture. *C.R. Acad. Sci. Paris B.* **287**, 209 (1978)
58. Garcia, R., Chan, M.H.W.: Critical Casimir effect near the  $^3\text{He}$ - $^4\text{He}$  tricritical point. *Phys. Rev. Lett.* **88**, 086101 (1978)
59. Hertlein, C., Helden, L., Gambassi, A., Dietrich, S., Bechinger, C.: Direct measurement of critical Casimir forces. *Nature* **451**, 172 (2008)