Calculations for Nondestructive BEC Imaging

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1 Diffraction by a Phase Object

We describe the phase object (BEC) by the deviation $\delta n(x,y,z)$ of the local index $n(x,y,z)$ from unity:

$$\delta n(x,y,z) := n(x,y,z) - 1. \tag{1}$$

In the thin-lens approximation, we can integrate over the propagation direction (z) of the probe, and consider only the 2-dimensional index profile:

$$\delta n_z(x,y) := \int \delta n(x,y,z)\, dz. \tag{2}$$

This phase shift imparts momentum to the light as follows: in the scalar-wave approximation, the mean-square transverse wave vector $\langle k_\mathrm{T}^2\rangle$ due to the relative phase shift of $\exp[ik\delta n_z(x,y)]$ (k is the total wave vector) is

$$\begin{aligned}\langle k_\mathrm{T}^2\rangle &= \frac{1}{|E|^2}\int E^*(x,y)(k_x^2+k_y^2)E(x,y)\,dx\,dy\\ &= -\int e^{-ik\delta n_z}(\partial_x^2+\partial_y^2)e^{ik\delta n_z}\,dx\,dy\\ &= k^2\int \left[(\partial_x\delta n_z)^2+(\partial_y\delta n_z)^2\right]\,dx\,dy.\end{aligned} \tag{3}$$

Here we have dropped terms of the form $\partial_x^2\delta n_z$, which do not contribute if we assume that δn_z is purely real (i.e., we ignore absorption effects).

To make this expression more concrete, we can consider a Gaussian approximation to the true BEC atomic distribution,

$$\delta n_z = \frac{\delta\phi_\mathrm{a}}{2\pi\sigma_x\sigma_y}\exp\left[-\frac{x^2}{2\sigma_x^2}-\frac{y^2}{2\sigma_y^2}\right], \tag{4}$$

where $\delta\phi_\mathrm{a}$ is the integrated phase shift:

$$\delta\phi_\mathrm{a} := \int \delta n(x,y,z)\,dx\,dy\,dz. \tag{5}$$

Then we can write

$$\langle k_\mathrm{T}^2\rangle = \frac{(\delta\phi_\mathrm{a})^2(\sigma_x^2+\sigma_y^2)k^2}{8\pi\sigma_x^3\sigma_y^3} \tag{6}$$

for the optical transverse momentum in the Gaussian approximation.

This momentum is similarly imparted to the atomic cloud, and the atomic momentum diffusion is given by $\langle \hbar^2 k_{\mathrm{T}}^2 \rangle$ multiplied by the incident photon flux. Thus we can write

$$D = \frac{I\hbar}{N\omega} \langle k_{\mathrm{T}}^2 \rangle \tag{7}$$

for the momentum diffusion rate per atom, where I is the probe intensity, ω is the optical frequency, and N is the number of atoms in the condensate.

2 Atomic Parameters

We can write the local index as (see, for example, Cohen-Tannoudji, Dupont-Roc, and Grynberg, *Atom-Photon Interactions*, p. 604)

$$\delta n = -\frac{n_{\mathrm{d}} |d|^2}{2\epsilon_0 \hbar} \frac{\Delta}{\Delta^2 + \Gamma^2/4 + \Omega^2/2}, \tag{8}$$

where n_{d} is the local number density, d is the appropriate dipole moment (here, approximately the effective dipole moment for far-detuned, linearly polarized light, see "Rubidium 87 D Line Data" for details). In the far-detuned limit, the integrated phase shift is then

$$\delta\phi_{\mathrm{a}} = -\frac{N|d|^2}{2\epsilon_0 \hbar \Delta}. \tag{9}$$

Thus the diffusion rate in the Gaussian approximation is

$$D = (\hbar^2 k^2) \frac{N|d|^4}{4\epsilon_0^2 \hbar^3 \omega} \frac{I}{\Delta^2} \frac{\sigma_x^2 + \sigma_y^2}{8\pi \sigma_x^3 \sigma_y^3}, \tag{10}$$

which has an overall scaling similar to spontaneous scattering, I/Δ^2.

3 Signal/Noise

From the Kadlecek et al. paper, the signal/noise ratio is given by

$$\mathrm{S/N} = \phi \sqrt{2\eta N_{\mathrm{p}}}, \tag{11}$$

where N_{p} is the number of photons striking a particular CCD pixel, η is the CCD quantum efficiency, we consider only the limit of large reference-beam intensity, and we have reduced the value quoted here by a factor of $\sqrt{2}$ from the value quoted in the paper, as is appropriate for spatial-heterodyne mode (since we must average over the heterodyne phase in the mean-square sense). This expression is also valid only for small local phase shifts ϕ; for large ϕ, we must average also over ϕ, and in this case the signal/noise is given by the same expression, but with ϕ replaced by 1. To evaluate the signal/noise ratio, we simply note that the typical phase shift is

$$\phi \approx -\frac{N|d|^2 k}{2\epsilon_0 \hbar \Delta \sigma_x \sigma_y}, \tag{12}$$

and the photon number is

$$N_{\mathrm{p}} = \frac{I t_{\mathrm{exp}} A_{\mathrm{pix}}}{\hbar\omega}, \tag{13}$$

where t_{exp} is the exposure time of the image, and A_{pix} is the effective CCD pixel area, taking into account any magnification in the imaging system. Thus in the small ϕ regime, where the experiment is likely to operate, the signal/noise ratio scales as $\sqrt{I}/\Delta^2$.

$\mathrm{Hz} := \frac{1}{\mathrm{sec}}$ $\mathrm{MHz} := 10^{6}\cdot\mathrm{Hz}$ $\mathrm{kHz} := 10^{3}\cdot\mathrm{Hz}$ $\mathrm{GHz} := 10^{9}\cdot\mathrm{Hz}$ $\mathrm{THz} := 10^{12}\cdot\mathrm{Hz}$

$\mathrm{mW} := 10^{-3}\cdot\mathrm{watt}$ $\mathrm{ms} := 10^{-3}\cdot\mathrm{sec}$ $\mathrm{cm} := 10^{-2}\cdot\mathrm{m}$

$\mathrm{hbar} := 1.054571596\cdot 10^{-34}\cdot\mathrm{joule}\cdot\mathrm{sec}$ $\varepsilon 0 := 8.854187817\cdot 10^{-12}\cdot\frac{\mathrm{farad}}{\mathrm{m}}$ $\mu\mathrm{m} := 10^{-6}\cdot\mathrm{m}$

Parameters for Rb 87

$c := 2.99792458\cdot 10^{8}\cdot\frac{\mathrm{m}}{\mathrm{sec}}$ $\omega := 2\cdot\pi\cdot 384.2279818773\cdot\mathrm{THz}$ $\lambda := 2\cdot\pi\cdot\frac{c}{\omega}$ $k := \frac{2\cdot\pi}{\lambda}$

$\Gamma := 2\cdot\pi\cdot 6.065\cdot\mathrm{MHz}$ $d := 2.069\cdot 10^{-29}\cdot\mathrm{coul}\cdot\mathrm{m}$

$N := 2\cdot 10^{5}$ $\sigma x := 100\cdot\mu\mathrm{m}$ $\sigma y := \sigma x$ $\mathrm{Isat} := \frac{c\cdot\varepsilon 0\cdot\Gamma^{2}\cdot\mathrm{hbar}^{2}}{4\cdot d^{2}}$ $\mathrm{Isat} = 2.504\cdot\frac{\mathrm{mW}}{\mathrm{cm}^{2}}$

$\Delta := 2\cdot\pi\cdot 2\cdot\mathrm{GHz}$ $I := 2\cdot\frac{\mathrm{mW}}{\mathrm{cm}^{2}}$

$\delta\phi a := \frac{-N\cdot d^{2}}{2\cdot\varepsilon 0\cdot\mathrm{hbar}\cdot\Delta}$ $\delta\phi a = -3.64826\cdot\mu\mathrm{m}^{3}$ $4\cdot\left(\frac{\Delta}{\Gamma}\right)^{2} = 4.35\cdot 10^{5}$

$\frac{I}{\mathrm{Isat}} = 0.799$

Diffusion due to coherent lensing

$$D := \frac{N\cdot d^{4}}{4\cdot\varepsilon 0^{2}\cdot\mathrm{hbar}^{3}\cdot\omega}\cdot\frac{I}{\Delta^{2}}\cdot\frac{\sigma x^{2}+\sigma y^{2}}{8\cdot\pi\cdot\sigma x^{3}\cdot\sigma y^{3}}\cdot\mathrm{hbar}^{2}\cdot k^{2}$$

$$D = 4.16\cdot 10^{-9}\cdot\frac{\mathrm{hbar}^{2}\cdot k^{2}}{\mathrm{ms}}$$

Diffusion due to spontaneous emission

$$\mathrm{Rsc} := \left(\frac{\Gamma}{2}\right)\cdot\left(\frac{I}{\mathrm{Isat}}\right)\cdot\left[1+4\cdot\left(\frac{\Delta}{\Gamma}\right)^{2}+\left(\frac{I}{\mathrm{Isat}}\right)\right]^{-1} \qquad \mathrm{Rsc} = 0.035\cdot\frac{1}{\mathrm{ms}}$$

$$\mathrm{Dsp} := \mathrm{Rsc}\cdot\mathrm{hbar}^{2}\cdot k^{2} \qquad \mathrm{Dsp} = 0.035\cdot\frac{\mathrm{hbar}^{2}\cdot k^{2}}{\mathrm{ms}}$$

Signal/noise ratio

$\phi := -\frac{N\cdot d^{2}\cdot k}{2\cdot\varepsilon 0\cdot\mathrm{hbar}\cdot\Delta\cdot\sigma x\cdot\sigma y}$ $\phi = -0.003$ $\mathrm{Apix} := (10\cdot\mu\mathrm{m})^{2}$

$\eta := 0.3$ $\mathrm{Np} := \frac{I\cdot\mathrm{Apix}}{\mathrm{hbar}\cdot\omega}$ $\mathrm{Np} = 7.856\cdot 10^{6}\cdot\frac{1}{\mathrm{ms}}$

$\mathrm{SN1} := |\phi|\cdot\sqrt{2\cdot\eta\cdot\mathrm{Np}}$ $\mathrm{SN1} = 6.378\cdot\frac{1}{\sqrt{\mathrm{ms}}}$ (valid for small phase shift)